Published by Institute of Physics Publishing for SISSA

Received: *May 3, 2008*
Revised: *July 9, 2008*
Accepted: *July 10, 2008*
Published: *July 25, 2008*

Dangerous Liouville wave — exactly marginal but non-conformal deformation

Chiu Man Ho

Department of Physics, University of California,
Berkeley, CA 94720 U.S.A.
E-mail: cmho@berkeley.edu

Yu Nakayama

Theoretical Physics Group, Lawrence Berkeley National Laboratory,
Berkeley, CA 94720 U.S.A.
E-mail: nakayama@berkeley.edu

Abstract: We give a non-trivially interacting field theory example of scale invariant but non-conformal field theory. The model is based on the exactly solvable Liouville field theory coupled with free scalars deformed by an exactly marginal operator. We show non-vanishing of the trace of the energy-momentum tensor by using the quantum Schwinger-Dyson equation for the Liouville field theory, which is a sophistication of the quantum higher equations of motion for the Liouville field theory introduced by Alyosha Zamolodchikov. Possibly dangerous implications for the super-critical string theory will be discussed.

Keywords: Field Theories in Lower Dimensions, Conformal Field Models in String Theory, Sigma Models.

JHEP07(2008)109

Contents

1. Introduction

The distinction between the scale invariance and conformal invariance was certainly an issue at the advent of the latter. According to a legend [1], when a provocative question about the difference between the two was addressed by a Western physicist at an international conference on scale invariance in Dubna, a great mathematician, who was a chairman at the session, literally said "There is no mathematical difference, but when some young people want to use a fancy word they call it Conformal Symmetry". A young brilliant physicist in the Soviet Union suddenly stood up and yelled "15 parameters!" but it echoed apparently unnoticed.

This issue is not so trivial, and the great mathematician was in some sense correct from the viewpoint of empirical science because we do not know any good physical examples of scale-invariant but non-conformal field theories in four-dimension. In two-dimension, his claim is even mathematically true because, as later discovered [2−4], one can give a proof of the equivalence between the two notions under certain conditions such as unitarity.

Today, the question whether the conformal symmetry is a fancy alternative word for the scale invariance is a hot topic in high energy phenomenology. Followed by a seminal work by H. Georgi [5], many works have been done to study a possible existence of a scale invariant (but non-conformal) hidden sector in our real world and experimental evidence for such "unparticle physics", which is spectacular in many cases. A very few authors have recognized the difference between the scale invariance and the conformal invariance in this context, and we have stressed the severe unitarity bound constraint coming from the latter in [6, 7]. Given the theoretical situation above, the experimental discovery of scale invariant but non-conformal "unparticle" would be a supreme surprise in theoretical physics.

As we mentioned, quantum examples of scale invariant but non-conformal field theory are very scarce (see e.g. [8] for a notable exception).[1] In this paper, we add a new two-dimensional example of such based on the Liouville field theory. The model is fully quantized by virtue of the exact solvability of the Liouville field theory. Although our model is not unitary as can be inferred from the general "proof" of the equivalence between scale invariance and conformal invariance in two-dimension, it may be applied to the world-sheet formulation of the perturbative string theory.

The organization of the paper is as follows. In section 2, we review the relation between scale invariance and conformal invariance from the viewpoint of conserved currents. In section 3, we introduce a class of classical examples of scale invariant but non-conformal field theories in two-dimension. In section 4, we investigate a quantum version of such a model based on the Liouville field theory. The quantum Schwinger-Dyson equation in the Liouville field theory, which is crucial to understand the violation of the conformal symmetry, is thoroughly studied. In section 5, we give some further discussions of our results.

2. Scale invariance vs conformal invariance

Einstein's special relativity suggests that a basic space-time symmetry of the quantum field theory (in d-dimension) is generated by the Poincare algebra:

$$\begin{aligned} i[J^{\mu\nu}, J^{\rho\sigma}] &= \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\sigma\mu} J^{\rho\nu} + \eta^{\sigma\nu} J^{\rho\mu} \\ i[P^{\mu}, J^{\rho\sigma}] &= \eta^{\mu\rho} P^{\sigma} - \eta^{\mu\sigma} P^{\rho} \\ [P^{\mu}, P^{\nu}] &= 0 \ . \end{aligned} \tag{2.1}$$

For massless scale invariant theory, one can augment this Poincare algebra by adding the dilatation operator D as

$$\begin{aligned} [P^{\mu}, D] &= iP^{\mu} \\ [J^{\mu\nu}, D] &= 0 \ . \end{aligned} \tag{2.2}$$

The generalization of Coleman-Mandula theorem [10] asserts (for $d \geq 3$) that the maximally enhanced bosonic symmetry of the space-time algebra for massless particles is given by the conformal algebra (plus some internal symmetries):

$$\begin{aligned} [K^{\mu}, D] &= -iK^{\mu} \\ [P^{\mu}, K^{\nu}] &= 2i\eta^{\mu\nu} D + 2iJ^{\mu\nu} \\ [K^{\mu}, K^{\nu}] &= 0 \\ [J^{\rho\sigma}, K^{\mu}] &= i\eta^{\mu\rho} K^{\sigma} - i\eta^{\mu\sigma} K^{\rho} \, , \end{aligned} \tag{2.3}$$

where K^{μ} generate special conformal transformation.

As is clear from the group theory structure above, the conformal symmetry demands the scale invariance but the reverse is not necessarily true: scale invariance does not always

[1] In [9], other classical examples of scale invariant field theory in four-dimension without conformal invariance are given. However, the scale invariance is spontaneously broken there.

imply conformal invariance.[2] A simple example of such theories with scale invariance but without conformal invariance is a free massless vector field with no gauge invariance [13, 14]. The two-dimensional massless vector field in this context was thoroughly investigated in [15].

However, in reality, every known unitary quantum scale invariant field theory in higher dimension than two is also conformal. The above-mentioned example of free massless vector field with no gauge invariance is not a unitary theory. Furthermore, one can even give a proof of the equivalence between the scale invariance and the conformal invariance for unitary theories with a discrete spectrum in two-dimension [4].

The distinction between the scale invariance and the conformal invariance in field theories can be summarized by the properties of the symmetric energy-momentum tensor $T_{\mu\nu}$. The dilatation current S^μ can be generated by

$$S^\mu = x_\nu T^{\nu\mu} + J^\mu \, , \tag{2.4}$$

where J^μ is a so-called virial current. Conservation of the dilatation current immediately implies

$$T^\mu_{\ \mu} = -\partial_\mu J^\mu \ . \tag{2.5}$$

Therefore, the necessary and sufficient condition of the scale invariance is that the energy-momentum tensor is a total divergence.

Moreover, if the virial current itself is a total derivative:

$$\begin{aligned} T^\mu_{\ \mu} &= \partial_\mu \partial_\nu L^{\mu\nu} \quad (d \geq 3) \\ &= \partial^\mu \partial_\mu L \quad\ \ (d = 2) \, , \end{aligned} \tag{2.6}$$

one can improve the energy-momentum tensor so that it is traceless (see e.g. [4] for details)

$$\Theta^\mu_{\ \ \mu} = 0 \ . \tag{2.7}$$

By using this improved traceless energy-momentum tensor, one can construct conserved currents

$$j^\mu_v = v_\nu \Theta^{\nu\mu} \, , \tag{2.8}$$

where the vector v^μ satisfies

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \frac{2}{d} \eta_{\mu\nu} \partial_\rho v^\rho \ . \tag{2.9}$$

The currents j^μ_v generate all the conformal transformation. In particular, one can obtain the special conformal transformation associated with K^μ by taking $v_\mu = \rho_\mu x^\nu x_\nu - 2x_\mu \rho^\nu x_\nu$, where ρ_μ is a constant vector parameter.

In this way, the study of the scale invariant but non-conformal field theory is reduced to the problem whether the virial current is a total derivative or not. In two-dimension, one can show that $\langle {\Theta^\mu_{\ \ \mu}}^\dagger \Theta^\mu_{\ \ \mu} \rangle = 0$ with scale invariance [2–4], which implies $\Theta^\mu_{\ \ \mu} = 0$ (conformal invariance) for a unitary and compact theory.

[2] See [11, 12] for earlier references on the interplay between scale invariance and conformal invariance.

3. Classical Liouville field theory with dangerous perturbation

Reference [14] showed a class of two-dimensional examples of classical field theories that are scale invariant but have no conformal invariance. The model is based on the classical Liouville field theory, so we would like to begin with a brief review of the conformal invariance of the classical Liouville field theory. The Liouville field theory[3] has the action

$$S_{\text{Liouville}} = \frac{1}{4\pi}\int d^2z \left(\partial^\mu\phi\partial_\mu\phi + 4\pi\mu e^{2b\phi}\right) , \tag{3.1}$$

where the classical limit corresponds to $b \to 0$.

The Liouville equation can be obtained as an equation of motion:

$$\partial_\mu\partial^\mu\phi = 4\pi\mu b e^{2b\phi} . \tag{3.2}$$

An energy-momentum tensor can be constructed from the Noether prescription:

$$T_{\mu\nu} = -\partial_\mu\phi\partial_\nu\phi + \frac{\eta_{\mu\nu}}{2}\left(\partial_\rho\phi\partial^\rho\phi + 4\pi\mu e^{2b\phi}\right) . \tag{3.3}$$

Of course, one could improve the energy-momentum tensor at this point by adding a total derivative $\partial_\mu\partial_\nu\phi - \eta_{\mu\nu}\partial_\rho\partial^\rho\phi$, but we do not do it here.

The trace of the energy-momentum tensor can be evaluated by using the equation of motion as

$$T^\mu_{\ \mu} = 4\pi\mu e^{2b\phi} = \frac{1}{b}\partial_\mu\partial^\mu\phi . \tag{3.4}$$

Thus, the virial current $J_\mu = -\frac{1}{b}\partial_\mu\phi$ is a total derivative, and the Liouville field theory is a conformal field theory. Indeed, one can construct the traceless energy-momentum tensor as

$$\Theta_{\mu\nu} = T_{\mu\nu} + \frac{1}{b}\left(\partial_\mu\partial_\nu\phi - \eta_{\mu\nu}\partial^\rho\partial_\rho\phi\right) , \tag{3.5}$$

which will yield a holomorphic energy-momentum tensor[4]

$$T(z) \equiv \Theta_{zz}(z) = -\partial\phi\partial\phi + \frac{1}{b}\partial^2\phi . \tag{3.6}$$

A class of classical scale invariant but non-conformal field theories is obtained [14] by coupling the Liouville field theory to a sigma model

$$S = \int d^2z\, G^{MN}(X^N)\partial_\mu X_M\partial^\mu X_N + S_{\text{Liouville}} + S_{\text{int}} \tag{3.7}$$

by the interaction

$$S_{\text{int}} = \frac{\lambda}{4\pi}\int d^2z\, h(X^N)\partial_\mu\phi\partial^\mu\phi \tag{3.8}$$

[3] We use the convention of [16].

[4] A quantum correction will modify the energy-momentum tensor as $T(z) = -\partial\phi\partial\phi + Q\partial^2\phi$, where $Q = b + b^{-1}$.

with a nontrivial scalar function $h(X^N)$ in the target space. The model is classically scale invariant with obvious scaling dimensions $D(X^N) = 0$ and $D(e^{2b\phi}) = 2$.

However, the model has no conformal invariance. To see this, let us compute the trace of the energy-momentum tensor:

$$T^{\mu}_{\ \mu} = 4\pi\mu e^{2b\phi} = \frac{1}{b}\partial_\mu \left[\left(1 + \lambda h(X^N) \right) \partial^\mu \phi \right] , \tag{3.9}$$

which is divergence of the virial current, and, as a consequence, the theory is expectedly scale invariant. However, the associated virial current

$$J_\mu = -\frac{1}{b} \left(1 + \lambda h(X^N) \right) \partial_\mu \phi \tag{3.10}$$

is not a total derivative for non-trivial $h(X^N)$, so the model is not a conformal field theory.

Before we go on constructing a quantum version of the above scale invariant but non-conformal field theory, several comments are in order.

- The Liouville interaction is crucial. For $\mu = 0$, one can recover the conformal invariance by setting $D(\phi) = 0$. Thus, exact treatment of the Liouville interaction would be needed when quantized.

- Quantum mechanically, one has to show that $h(X^N)$ has a non-trivial fixed point as well as the target space metric $G^{MN}(X^N)$. One-loop approximation will give you Einstein-dilaton equation coupled with the non-trivial tachyon. The Liouville interaction is very difficult to treat in this approach because it is strongly coupled, and higher α' corrections cannot be neglected. We take a different root to establish the fixed point in the next section.

- The model gives a "counterexample" for the proof of the equivalence between the scale invariance and conformal invariance in two-dimension. Assuming the nontrivial fixed point for $h(X^N)$, we see that the proof fails because of the non-compactness[5] of the target space (especially in the Liouville direction).

4. Quantum Liouville wave

In this section, we construct a concrete quantum example of scale invariant but non-conformal field theory based on the model presented in section 3. We take a sigma model as a flat target-space with signature $(1,1)$. The action is

$$S = \frac{1}{4\pi} \int d^2z \left(\partial^\mu X^1 \partial_\mu X^1 - \partial^\mu X^0 \partial_\mu X^0 \right) + S_{\text{Liouville}} + S_{\text{int}} . \tag{4.1}$$

[5]The non-compactness of the target space also played a crucial role in the examples of scale invariant but non-conformal field theories studied in [8] .

The interaction takes a form of the light-cone wave:[6]

$$S_{\text{int}} = \frac{\lambda}{4\pi} \int d^2 z \left(e^{iv(X^1 - X^0)} \partial^\mu \phi \partial_\mu \phi \right) \ . \tag{4.2}$$

In other words, we take $h = e^{iv(X^1 - X^0)}$.

As before, the trace of the (classical) energy-momentum tensor

$$T^\mu_{\ \mu} = \frac{1}{b} \partial^\mu \left[\left(1 + \lambda e^{iv(X^1 - X^0)} \right) \partial_\mu \phi \right] \tag{4.3}$$

cannot be improved to be zero. Alternatively, the formerly holomorphic energy-momentum tensor now becomes

$$T = -\partial X^1 \partial X^1 + \partial X^0 \partial X^0 - \left(1 + \lambda e^{iv(X^1 - X^0)} \right) \partial\phi \partial\phi + Q \partial^2 \phi \, , \tag{4.4}$$

which is classically no longer holomorphic

$$\bar{\partial} T = \frac{1}{b} \partial [\, \partial \bar{\partial} \phi - \pi \mu b e^{2b\phi} \,] \ \neq 0 \ . \tag{4.5}$$

As a consequence, to see the quantum mechanical violation of the conformal symmetry of this system, one can investigate the following correlation functions

$$\begin{aligned} & b \left\langle \, \bar{\partial} T(x_T) \ O_1 \cdots O_N \, \right\rangle \\ &= \left\langle \, \partial(\partial\bar{\partial}\phi - \pi\mu b e^{2b\phi})(x_T) \ O_1 \cdots O_N \, \right\rangle \\ &= \sum_n \frac{1}{n!} \left\langle \partial(\partial\bar{\partial}\phi - \pi\mu b e^{2b\phi})(x_T) \ O_1 \cdots O_N \left[\frac{-\lambda}{4\pi} \int d^2 z \, e^{iv(X^1 - X^0)} \partial_\mu \phi \partial^\mu \phi \right]^n \right\rangle_{\lambda=0} \end{aligned} \tag{4.6}$$

where O_i are inserted at $x = x_i$. We have neglected possible contact terms in the first equality, which do not play any role in the conformal symmetry breaking.[7] The second equality is a perturbative series evaluated by the unperturbed Liouville field path integral. Actually, the perturbative series is not a formal summation but contains only a single term for each set of O_i with fixed charge due to the charge conservation for X^1 and X^0. In the following, we will show that (4.6) does not vanish so that the conformal invariance is indeed violated quantum mechanically.

As a side remark, the first equality in (4.5) might seem to rely on the classical equation of motion and need possible quantum modifications in the evaluation of (4.6). However,

[6]The action is not hermitian with our choice of the interaction. However, since our discussion does not depend v as we will see in the following, one can formally perform analytic continuation $v \to iv$ to make the action hermitian. See also footnote 13 for a related point. In any case, we do not require unitarity, so this is not a primary concern of our construction.

[7]Note that the conformal invariance (or breaking) does not say anything about the structure of the contact terms. What is relevant in the following, however, is that for non-zero λ, we have to integrate the additional contact terms over the inserted position to obtain non-zero non-contact terms that will break conformal invariance.

JHEP07(2008)109

the Liouville equation of motion is exact, so for $\lambda = 0$, we do not need any modification.[8] As a perturbative quantum expansion in λ, order by order quantum redefinition of the energy-momentum tensor does not recover the holomorphicity because it is broken at the classical level and the quantum modification cannot compensate the classical piece, as long as the classical equation of motion is compatible with the exact quantization as we will show explicitly.

Even with the Liouville equation of motion $\partial\bar{\partial}\phi - \pi\mu b e^{2b\phi} = 0$ for the unperturbed action ($\lambda = 0$), the series (4.6) does not generically vanish. The quantum equation of motion (Schwinger-Dyson equation) possesses a contact term at $x_i = x_T$:

$$\frac{2}{\pi}\left\langle (\partial\bar{\partial}\phi - \pi\mu b e^{2b\phi})(x_T)\ O_1 \cdots O_N \right\rangle_{\lambda=0} = \sum_i \left\langle O_1 \cdots \frac{\delta O_i(x_i)}{\delta\phi(x_T)} O_N \right\rangle_{\lambda=0} . \quad (4.7)$$

Formally, one can obtain the Schwinger-Dyson equation from the invariance of the path integral measure

$$\begin{aligned} \int \mathcal{D}\phi\ O_1 \cdots O_N\, e^{-S} &= \int \mathcal{D}(\phi + \delta\phi)\ O_1 \cdots O_N\, e^{-S} \\ \Longleftrightarrow 0 &= \int \mathcal{D}\phi\ \frac{\delta}{\delta\phi}\left(O_1 \cdots O_N\, e^{-S}\right) . \end{aligned} \quad (4.8)$$

The contact terms in the Schwinger-Dyson equation at $z = x_T$ after integrating over the inserted position z will give you the failure of the holomorphicity of the energy-momentum tensor in (4.6).

In the following, we focus on the contact terms in the Liouville equation of motion in the Liouville correlation functions denoted by $\langle \cdots \rangle_L$ among the Liouville primary vertex operators $V_\alpha \sim e^{2\alpha\phi}$. From the path integral argument, we expect the following identity:

$$\begin{aligned} &\frac{2}{\pi}\left\langle (\partial\bar{\partial}\phi - \pi\mu b e^{2b\phi})(x_T)\ e^{2\alpha_1\phi(x_1)} \cdots e^{2\alpha_N\phi(x_N)} \right\rangle_L \\ &= \sum_i 2\alpha_i \delta(x_i - x_T)\left\langle e^{2\alpha_1\phi(x_1)} \cdots e^{2\alpha_N\phi(x_N)} \right\rangle_L . \end{aligned} \quad (4.9)$$

The quantum treatment of the higher equations of motion in Liouville field theory was initiated in [17] (see also [18] for a subsequent work). We first introduce the logarithmic primary operator:

$$V_0' = \frac{1}{2}\frac{\partial}{\partial\alpha} V_\alpha|_{\alpha=0} \simeq \phi . \quad (4.10)$$

Then, [17] showed that the correlation function is invariant under the replacement (we recall $L_{-1} = \partial$)

$$L_{-1}\bar{L}_{-1}V_0' = \pi\mu b V_{\alpha=b} . \quad (4.11)$$

[8]The only exception is the case where $v = 0$. In this case, the contact term between the Liouville equation of motion and the perturbative interaction after integration, which is actually nothing but a deliberate separation of the Liouville kinetic term, gives you contribution $\bar{\partial}T = -\frac{1}{b}\partial(\lambda\partial\bar{\partial}\phi)$. This can be absorbed by a redefinition of the holomorphic energy-momentum tensor as $T \to T + \frac{\lambda}{b}\partial^2\phi$. Note that this redefinition cannot be done for non-zero v even classically.

The derivation of [17] is only valid up to contact terms. We now show a refinement of his argument to derive the contact term contributions to the quantum equation of motion.

As in [17], we concentrate on the three-point function

$$\left\langle L_{-1}\bar{L}_{-1}V'_\alpha(x_T)V_{\alpha_1}(x_1)V_{\alpha_2}(x_2)\right\rangle_L \tag{4.12}$$

and study $\alpha \to 0$ limit. The three-point function takes the form

$$\begin{aligned}&\left\langle V'_\alpha(x_T)V_{\alpha_1}(x_1)V_{\alpha_2}(x_2)\right\rangle_L \\ =&\frac{1}{2}\frac{\partial}{\partial\alpha}\left[\frac{C(\alpha,\alpha_1,\alpha_2)}{|x_1-x_2|^{2\Delta_1+2\Delta_2-2\Delta}|x_T-x_1|^{2\Delta_1+2\Delta-2\Delta_2}|x_T-x_2|^{2\Delta_2+2\Delta-2\Delta_1}}\right],\end{aligned} \tag{4.13}$$

where the conformal weight of the Liouville primary operator V_α is given by $\Delta=\alpha(Q-\alpha)$. The structure constant $C(\alpha,\alpha_1,\alpha_2)$ of the Liouville field theory was computed [19–21] to be

$$\begin{aligned}C(\alpha,\alpha_1,\alpha_2)=&[\pi\mu\gamma(b^2)b^{2-2b^2}]^{(Q-\alpha-\alpha_1-\alpha_2)/b}\\ &\times\frac{\Upsilon'(0)\Upsilon(2\alpha)\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)}{\Upsilon(\alpha+\alpha_1+\alpha_2-Q)\Upsilon(\alpha+\alpha_1-\alpha_2)\Upsilon(\alpha_1+\alpha_2-\alpha)\Upsilon(\alpha_2+\alpha-\alpha_1)},\end{aligned} \tag{4.14}$$

where $\Upsilon(x)$ is defined by

$$\log\Upsilon(x)=\int_0^\infty\frac{dt}{t}\left[\left(\frac{Q}{2}-x\right)^2e^{-t}-\frac{\sinh^2(\frac{Q}{2}-x)\frac{t}{2}}{\sinh\frac{bt}{2}\sinh\frac{t}{2b}}\right] \tag{4.15}$$

for $0<\mathrm{Re}(x)<Q$ and analytically continued to the whole complex plane. See e.g. [20, 16] for some properties of the special functions.

For generic value of α_1 and α_2, the structure constant $C(\alpha,\alpha_1,\alpha_2)$ has a simple zero as $\alpha\to 0$, and only the term with $\partial_\alpha C(\alpha,\alpha_1,\alpha_2)$ in (4.13) contributes as discussed in [17]. This is consistent with the contact term contribution that should yield like $\delta(x_T-x_1)\langle V_{\alpha_1}(x_1)V_{\alpha_2}(x_2)\rangle_L$, which is non-zero only in the $\alpha_1\to\alpha_2$ limit (or $\alpha_1\to Q-\alpha_2$ limit).

We, thus, take a careful limit of $\alpha\equiv\epsilon\to 0$ and $\alpha_1-\alpha_2\equiv i\kappa\to 0$.[9] In this limit, (4.13) becomes

$$\frac{\partial}{\partial\epsilon}\left[\frac{2\epsilon}{(\epsilon+i\kappa)(\epsilon-i\kappa)}\left(\frac{S(\alpha_1)}{|x_1-x_2|^{4\Delta_1-2\epsilon Q}|x_1-x_T|^{2Q\epsilon}|x_2-x_T|^{2Q\epsilon}}+\mathcal{O}(\epsilon,\kappa)\right)\right], \tag{4.16}$$

where $S(\alpha_1)$ is the two-point function of the Liouville field theory: $\langle V_{\alpha_1}(1)V_{\alpha_2}(0)\rangle_L = S(\alpha_1)\pi\delta(i\alpha_1-i\alpha_2)+\pi\delta(i\alpha_1+i\alpha_2-iQ)$, whose explicit form is given by

$$S(\alpha)=\frac{(\pi\mu\gamma\, b^2)^{(Q-2\alpha)/b}}{b^2}\frac{\gamma(2\alpha b-b^2)}{\gamma(2-2\alpha/b-1/b^2)}, \tag{4.17}$$

where $\gamma(x)=\frac{\Gamma(x)}{\Gamma(1-x)}$. We regard the first factor in (4.16) as the delta-function: $\lim_{\epsilon\to 0}\frac{2\epsilon}{(\epsilon+i\kappa)(\epsilon-i\kappa)}=2\pi\delta(\kappa)$. Then, the derivative with respect to ϵ gives the logarithmic term

$$2\pi\delta(i\alpha_1-i\alpha_2)S(\alpha_1)|x_1-x_2|^{-4\Delta_1}2Q\left(\log|x_1-x_T|+\log|x_2-x_T|\right)+\cdots, \tag{4.18}$$

[9] The reason for i in $\alpha_1-\alpha_2$ is that we take the physical normalizable Liouville momenta: $\alpha\in\frac{Q}{2}+i\mathbf{R}$.

where the ellipsis contains only x_T independent terms.

We take the laplacian of (4.18) with x_T from the insertion of $L_{-1}\bar{L}_{-1} = \partial\bar{\partial}$. By using the formula $\partial\bar{\partial}\log|z|^2 = \pi\delta(z)$, we obtain the sought-after contact term:

$$\begin{aligned}&\left\langle (L_{-1}\bar{L}_{-1}V_0' - \pi\mu b V_b)(x_T)\ V_{\alpha_1}(x_1)V_{\alpha_2}(x_2) \right\rangle_L \\ =&2\pi\delta(i\alpha_1 - i\alpha_2)S(\alpha_1)|x_1-x_2|^{-4\Delta_1}\pi Q\left(\delta(x_1-x_T)+\delta(x_2-x_T)\right) \ . \end{aligned} \tag{4.19}$$

In this way, we have shown that the contact terms indeed exist in the exact Liouville equation of motion, and from (4.6), we can now prove that the conformal invariance is broken for nonzero λ in the exact quantization of our model. In particular, note that the operator $\partial^\mu\phi\partial_\mu\phi$ inserted in (4.6) can be realized as a specific limit of the Liouville primary operator: $\partial^\mu\phi\partial_\mu\phi = 4 : L_{-1}V_0'\bar{L}_{-1}V_0' :$.

Another suggestive but not complete way to understand the importance of the contact terms in the conformal symmetry breaking is to perform partial integration inside the perturbative deformation to study the insertion of $\int d^2z\phi\partial\bar{\partial}\phi$. By using the undeformed Liouville equation of motion, it is equivalent to the insertion of $\int d^2z\phi e^{2b\phi} = \frac{1}{2}\int d^2z\frac{\partial}{\partial\alpha}V_\alpha|_{\alpha=b}$. The above computation directly shows that there exist contact terms for the vertex insertion $\frac{\partial}{\partial\alpha}V_\alpha|_{\alpha=b}$, and the integration over the inserted position z gives a non-contact term contribution to the energy-momentum tensor insertion.[10]

Nevertheless, the limiting procedure is a little bit subtle and one might claim an objection to the above derivation especially because (4.19) is different from the Schwinger-Dyson equation from the naive path integral (4.9). However, the naive Schwinger-Dyson equation (4.9) cannot be correct for the exact Liouville correlation function among V_α. It is in contradiction with the reflection symmetry [20] of the Liouville field theory: $V_\alpha \sim S(\alpha)V_{Q-\alpha}$.

To see this, suppose $V_\alpha = e^{2\alpha\phi}$ and use the naive Schwinger-Dyson equation (4.9):

$$\frac{2}{\pi}\left\langle (L_{-1}\bar{L}_{-1}V_0' - \pi\mu b V_b)(x_T)\ V_{\alpha_1}\cdots V_{\alpha_N} \right\rangle_L = 2\alpha_1\delta(x_T-x_1)\langle V_{\alpha_1}\cdots V_{\alpha_N}\rangle_L + \cdots \tag{4.20}$$

Alternatively, one could replace V_{α_1} with $S(\alpha_1)V_{Q-\alpha_1}$, and use the Schwinger-Dyson equation, and then replace $V_{Q-\alpha_1}$ with $S(\alpha_1)^{-1}V_{\alpha_1}$:

$$\begin{aligned}&\frac{2}{\pi}\left\langle (L_{-1}\bar{L}_{-1}V_0' - \pi\mu b V_b)(x_T)\ V_{\alpha_1}\cdots V_{\alpha_N} \right\rangle_L \\ =&\frac{2}{\pi}S(\alpha_1)\left\langle (L_{-1}\bar{L}_{-1}V_0' - \pi\mu b V_b)(x_T)\ V_{Q-\alpha_1}\cdots V_{\alpha_N} \right\rangle_L \\ =&2(Q-\alpha_1)\delta(x_T-x_1)\langle V_{\alpha_1}\cdots V_{\alpha_N}\rangle_L + \cdots \ , \end{aligned} \tag{4.21}$$

which is in contradiction with (4.20).

Any α dependence in the contact term is inconsistent with the reflection symmetry of the quantum Liouville field theory. The limiting procedure we showed in the above is the most natural one consistent with the reflection symmetry. Indeed, the discussion

[10] We should note, however, that there is an additional contribution $\int d^2z\partial^\mu\left[e^{iv(X^1-X^0)}\right]\phi\partial_\mu\phi$ which cannot be computed in this approach.

here suggests a deep insight about the Liouville primary vertex operator V_α. It seems quite plausible that the classical interpretation of V_α is not $e^{2\alpha\phi}$, but rather a mixture $e^{2\alpha\phi}+S(\alpha)e^{2(Q-\alpha)\phi}+\cdots$.[11] With the interim substitution of $V_\alpha \sim e^{2\alpha\phi}+S(\alpha)e^{2(Q-\alpha)\phi}$, the path integral approach in (4.9) agrees with the exact Schwinger-Dyson equation obtained from the exact three-point function with our limiting procedure.

4.1 Scale invariance

So far, we have discussed that the conformal symmetry is broken due to the coupling between the Liouville sector and the free boson sector. Even quantum mechanically, the Schwinger-Dyson equation of the Liouville field theory demands that the holomorphy of the energy-momentum tensor is violated. Now the question is whether the scale invariance is disturbed by this perturbation quantum mechanically. We would like to show some arguments that the interaction (4.2) is exactly marginal in the sense that the scale invariance is preserved.

First of all, as a necessary condition, our interaction Lagrangian has a quantum scaling dimension $D = 2$, which gives a first order perturbative condition for the scale invariance of the theory. To see higher order corrections, one can focus on the partition function

$$
\begin{aligned}
Z_\lambda &= \int \mathcal{D}\phi\mathcal{D}X^1\mathcal{D}X^0 \; e^{-S} \\
&= \sum_n \frac{1}{n!}\left\langle\left(\frac{-\lambda}{4\pi}\int d^2z \; e^{iv(X^1-X^0)}\partial_\mu\phi\partial^\mu\phi\right)^n\right\rangle_{\lambda=0} \\
&= Z_{\lambda=0} \; . \qquad (4.22)
\end{aligned}
$$

The last equality is due to the charge conservation. From this formal expression, one might naively conclude that, according to the general recipe of the conformal perturbation theory, we would not introduce any regularization or cut-off dependence, and hence the higher order beta function vanishes because the perturbative expansion of the partition function itself vanishes. However, in order to evaluate the beta function, what one has to really study is the singularity structure of the operator product expansions (OPEs) inside the formally vanishing perturbative corrections to the partition function that could be non-zero by adding background charges at infinity.

To address this question, we take a closer look at the singularity structure of the of OPEs of the Liouville sector and the sigma model sector separately. Firstly, in the Liouville sector, it is crucial to notice that the operator $\partial_\mu\phi\partial^\mu\phi$ is an exactly marginal deformation to the Liouville field theory: it just changes the normalization of the kinetic term.[12] This guarantees that there are no singular terms that cannot be absorbed by the field re-definition in the Liouville OPE from such deformation. More formally, one could define $\partial_\mu\phi\partial^\mu\phi$ as $4 : L_{-1}V_0'\bar{L}_{-1}V_0' :$ in the abstract Liouville field theory language, and

[11] This can be also inferred from the analysis of the mini-superspace reflection amplitudes [20].

[12] One should note that because of the changes of the normalization of the kinetic term, the deformation *does* change the central charge of the Liouville field theory through the Fradkin-Tseytlin counter term [22, 23] $\delta Q\phi R$, which vanishes on the flat Euclidean space we are using. The non-compactness of the target-space, however, makes the deformation exactly marginal by avoiding the c-theorem [3].

JHEP07(2008)109

study the OPE. To evaluate the OPE among $\partial_\mu\phi\partial^\mu\phi$, one can first investigate the OPE among the Logarithmic primary operators:

$$\begin{aligned} L_{-1}V_0'(z)\cdot\bar{L}_{-1}V_0'(0) &\sim \frac{S(b)}{|z|^2} \\ L_{-1}V_0'(z)\cdot L_{-1}V_0'(0) &\sim \frac{1}{z^2}+\frac{S(b)\log(\bar{z})}{z^2}\,, \end{aligned} \tag{4.23}$$

and so on. Note that $L_{-1}V_0'$ (or $\bar{L}_{-1}V_0'$) is no-longer a left (right) moving primary operator but still is a right (left) moving primary operator [17]. One can now see that the leading OPE singularity among $\partial_\mu\phi\partial^\mu\phi$ is exactly the same as that for the free scalar field theory, which means that the addition of the term simply changes the normalization of the kinetic term of the Liouville field, as in the free scalar field theory. The additional logarithmic term should be renormalized by the Fradkin-Tseytlin counter term, which is indeed necessary to keep the scale invariance even in the Liouville theory with no deformation (e.g. Polyakov regularization [24] gives $\lim_{w\to z}\log|w-z|^2 = -2\log|\rho(z)|^2$, where $\sqrt{g}R = -4\partial\bar{\partial}\log|\rho|^2$).

Secondly, in the sigma model sector, we note the fact that the light-cone scalar is non-singular in its OPE, namely $(X^1-X^0)(z)\cdot(X^1-X^0)(0)\sim 0$ which implies $e^{iu(X^1-X^0)}(z)\cdot e^{iv(X^1-X^0)}(0)\sim e^{i(u+v)(X^1-X^0)}(0)$, suggests that there are actually no additional singular contributions to the whole perturbation series.

Combining all these two sectors together, we have no hidden cut-off dependence in the partition function (even with background charge), and, therefore, we preserve the scale invariance under the perturbation to all oder in λ.[13] Of course, some correlation functions are modified and operators acquire extra anomalous dimension matrices, but they should be renormalized independently of the beta function.

5. Discussion

In this paper, we have shown an example of scale invariant but non-conformal quantum field theories in two dimension. From the general argument [4], such a theory should be non-compact or non-unitary. In our case, the theory is both non-compact and non-unitary. The former is due to the Liouville direction and the latter is due to the time-like direction in the sigma model. Indeed, the correlation function of the trace of the (improved) energy-momentum tensor

$$\left\langle {\Theta^\mu{}_\mu}^\dagger \Theta^\mu{}_\mu \right\rangle \tag{5.1}$$

vanishes due to the charge conservation,[14] while the trace itself does not vanish as we have seen in the previous section. The failure of the proof in [4] here is due to this non-unitary nature of the correlation functions, which manifests itself as the lack of reflection positivity.

[13] The argument here actually suggests that a broader class of non-conformal but scale invariant field theories be obtained by choosing arbitrary left-moving function $h(X^N) = h(X^1-X^0)$.

[14] The argument is as follows. We set $\Theta^\mu{}_\mu = -\frac{4}{b}\partial\bar{\partial}\phi + 4\pi\mu e^{2b\phi}$. The perturbative computation with respect to λ should be done at $\lambda=0$ because of the charge conservation. Then, the correlation function vanishes due to the Liouville equation of motion.

Although our model might have no physical significance as a two dimensional field theory because of the lack of the unitarity, it may have some applications in string theory, where the world-sheet theory needs not be unitary as long as ghosts are removed by the BRST constraint. From the viewpoint of the string worldsheet perturbation theory, this kind of exactly marginal but non-conformal deformation would be quite dangerous because it induces a world-sheet conformal anomaly, and it would lead to a potential swampland from the target-space viewpoint. Fortunately, the central charge of the Liouville sector is $c_\phi = 1 + 6(b + b^{-1})^2 \geq 25$, and the two extra dimensions for X^1 and X^0 make it difficult to embed our models in the critical string theory.

This dangerous situation could occur in the super-critical string theory (see e.g. [25, 26] and references therein), where we can introduce the time-like linear dilaton as well to reduce the central charge of the X^0 scalar as $c_{X^0} = 1-6(\beta-\beta^{-1})^2$ where β is the slope of the time-like linear dilaton. The world-sheet perturbation $e^{(2i\omega-(\beta-\beta^{-1}))X^0-2ikX^1}\partial\phi\bar{\partial}\phi$ could be an exactly marginal deformation (Liouville wave) under the condition $-\frac{(\beta-\beta^{-1})^2}{4}-\omega^2+k^2=0$. If the perturbation is exactly marginal, such a background would be inconsistent as a string background although the scale invariance is intact. It would be very interesting to see whether the Liouville wave deformation is possible within the super-critical string theory and investigate a possibly critical consequence of such a dangerous deformation.

Acknowledgments

One of the authors Y. N. would like to dedicate the paper to the memory of Alyosha Zamolodchikov, who shared the author with his enthusiasm in the Liouville field theory and his higher equations of motion. The research of Y. N. is supported in part by NSF grant PHY-0555662 and the UC Berkeley Center for Theoretical Physics. C. M. Ho acknowledges the support from the Croucher Foundation and Berkeley CTP.

References

[1] A. Migdal, *Ancient history of CFT*, in Proceedings of *The* 12$^{\text{th}}$ *Claude Itzykson meeting*, Saclay France June 18–22 2007.

[2] M. Lüscher and G. Mack, *The energy momentum tensor of a critical quan-tum field theory*, unpublished (1976);
G. Mack, *Introduction to conformal invariant quantum field theory in two or more dimensions*, in Proceedings of *NATO advanced summer institute on nonperturbative quantum field theory*, Cargese France July 16–30 1987, DESY-88-120.

[3] A.B. Zamolodchikov, *Irreversibility of the flux of the renormalization group in a* 2*D field theory*, *JETP Lett.* **43** (1986) 730 [*Pisma Zh. Eksp. Teor. Fiz.* **43** (1986) 565].

[4] J. Polchinski, *Scale and conformal invariance in quantum field theory*, *Nucl. Phys.* **B 303** (1988) 226.

[5] H. Georgi, *Unparticle physics*, *Phys. Rev. Lett.* **98** (2007) 221601 [hep-ph/0703260]; *Another odd thing about unparticle physics*, *Phys. Lett.* **B 650** (2007) 275 [arXiv:0704.2457].

[6] Y. Nakayama, *SUSY unparticle and conformal sequestering*, *Phys. Rev.* **D 76** (2007) 105009 [arXiv:0707.2451].

JHEP07(2008)109

[7] B. Grinstein, K. Intriligator and I.Z. Rothstein, *Comments on unparticles*, *Phys. Lett.* **B 662** (2008) 367 [arXiv:0801.1140].

[8] C.M. Hull and P.K. Townsend, *Finiteness and conformal invariance in nonlinear sigma models*, *Nucl. Phys.* **B 274** (1986) 349.

[9] S.B. Treiman, E. Witten, R. Jackiw and B. Zumino, *Current algebra and anomalies*, World Scientific, Singapore (1985).

[10] R. Haag, J.T. Lopuszanski and M. Sohnius, *All possible generators of supersymmetries of the S matrix*, *Nucl. Phys.* **B 88** (1975) 257.

[11] J. Wess, *The conformal invariance in quantum field theory*, *Nuovo Cim.* **18** (1960) 1086.

[12] G. Mack, *Partially conserved dilatation current*, Ph.D. thesis, Bern Switzerland (1967); G. Mack and A. Salam, *Finite component field representations of the conformal group*, *Ann. Phys. (NY)* **53** (1969) 174.

[13] S.R. Coleman and R. Jackiw, *Why dilatation generators do not generate dilatations?*, *Ann. Phys. (NY)* **67** (1971) 552.

[14] A. Iorio, L. O'Raifeartaigh, I. Sachs and C. Wiesendanger, *Weyl gauging and conformal invariance*, *Nucl. Phys.* **B 495** (1997) 433 [hep-th/9607110].

[15] V. Riva and J.L. Cardy, *Scale and conformal invariance in field theory: a physical counterexample*, *Phys. Lett.* **B 622** (2005) 339 [hep-th/0504197].

[16] Y. Nakayama, *Liouville field theory: a decade after the revolution*, *Int. J. Mod. Phys.* **A 19** (2004) 2771 [hep-th/0402009].

[17] A. Zamolodchikov, *Higher equations of motion in Liouville field theory*, *Int. J. Mod. Phys.* **A 19S2** (2004) 510 [hep-th/0312279].

[18] G. Bertoldi, S. Bolognesi, G. Giribet, M. Matone and Y. Nakayama, *Zamolodchikov relations and Liouville hierarchy in* $\mathrm{SL}(2,R)(k)$ *WZNW model*, *Nucl. Phys.* **B 709** (2005) 522 [hep-th/0409227].

[19] H. Dorn and H.J. Otto, *Two and three point functions in Liouville theory*, *Nucl. Phys.* **B 429** (1994) 375 [hep-th/9403141].

[20] A.B. Zamolodchikov and A.B. Zamolodchikov, *Structure constants and conformal bootstrap in Liouville field theory*, *Nucl. Phys.* **B 477** (1996) 577 [hep-th/9506136].

[21] J. Teschner, *On the Liouville three point function*, *Phys. Lett.* **B 363** (1995) 65 [hep-th/9507109].

[22] E.S. Fradkin and A.A. Tseytlin, *Effective field theory from quantized strings*, *Phys. Lett.* **B 158** (1985) 316.

[23] E.S. Fradkin and A.A. Tseytlin, *Quantum string theory effective action*, *Nucl. Phys.* **B 261** (1985) 1.

[24] A.M. Polyakov, *Quantum geometry of bosonic strings*, *Phys. Lett.* **B 103** (1981) 207.

[25] S. Hellerman and I. Swanson, *Cosmological solutions of supercritical string theory*, *Phys. Rev.* **D 77** (2008) 126011 [hep-th/0611317].

[26] S. Hellerman and I. Swanson, *Charting the landscape of supercritical string theory*, *Phys. Rev. Lett.* **99** (2007) 171601 [arXiv:0705.0980].

JHEP07(2008)109